

HOW TO CONSTRUCT METRICS TO CONTROL DISTRIBUTIONS OF HOMOLOGICALLY MASS-MINIMIZING CURRENTS

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ABSTRACT. By Federer and Fleming [FF60] there exist at least one mass-minimizing normal current in every real-valued homology class of a Riemannian manifold. However the regularity of the mass-minimizing currents and their distributions may generally be quite complicated. In this paper we shall study how to construct nice metrics so that (as functionals over smooth forms) almost all homologically mass-minimizing currents are just linear combinations over smooth submanifolds.

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1. INTRODUCTION

A calibration on a Riemannian manifold (see §2.1) is a closed differential m -form ϕ whose value at every point on every unit m -plane is at most one. The fundamental theorem of calibrated geometry in [HL82b] asserts that a smooth m -dimensional oriented submanifold (or more generally an integral or normal current) without boundary for which ϕ has value one at almost every point on every (generalized) unit tangent plane is mass-minimizing in its homology class of the complex of normal currents. Therefore the theory of calibrations is a powerful tool for proving the property of homological mass-minimality.

Notice that besides a background smooth manifold there are two extra slots, namely a balanced pair of a closed form and a metric, in a calibrated manifold. In this paper we shall create such balanced pairs for various situations. Particular interests are focused on two kinds of metric constructions based upon any given metric. One is the horizontal change in §3 and the other is the conformal change in §4. Given a homologically nontrivial, oriented, compact, connected smooth submanifold M , we show that one can alter any metric by the horizontal change or the

Date: January 9, 2015.

2010 Mathematics Subject Classification. 53C38, 49Q15, 28A75.

Key words and phrases. metric, homologically mass-minimizing current, calibration, comass, global Plateau property.

conformal change such that M becomes homologically mass-minimizing with respect to the resulted metric. It should be pointed out that Tasaki [Tas93] first proved the existence of metrics making M homologically mass-minimizing. Although the difference between Tasaki's result and ours reflects in the metric slot, our improvement essentially comes from (the construction of potential) calibrations. Lemma 2.15 provides us with a well-behaved potential local calibration, relying on which explicit metric gluing constructions can be made for global calibrations. The approach is feasible since submanifolds are "isolated" vs. a foliation (cf. [Glu], [Sul79] and [HL82a]).

The case of a constellation of mutually disjoint submanifolds possibly of different dimensions is also studied in §3 and §4. However only one calibration is constructed for each dimension. In order to control distributions of (almost all) homologically mass-minimizing currents we construct a metric under which enough calibrations are guaranteed for each dimension. Theorem 5.1 asserts that if X^n is oriented with betti numbers $b_k < \infty$ for $1 \leq k < n/2$, then there exists some metric in any conformal class such that all homologically area-minimizing currents of dimension $< n/2$ are currents of \mathbb{R} -weighted integration over mutually disjoint finitely many oriented, compact, connected submanifolds. By combination of the horizontal change and the conformal change, Theorem 5.2 confirms the general existence of such metrics allowing k to vary up to $n - 3$ (without the restriction to a fixed conformal class).

ACKNOWLEDGEMENT. This paper is an expansion of part of the author's Ph.D. thesis at Stony Brook University. He is deeply indebted to his advisor Professor H. Blaine Lawson, Jr. for suggesting this series of questions and constant encouragement. He also would like to thank Professor Frank Morgan and Professor Dennis Sullivan for helpful comments. Part of the work was polished during the author's visit to the MSRI in Fall 2013.

2. PRELIMINARIES

2.1. Calibrated Geometry. Let us review some fundamental concepts and results that we will need in this paper. Readers are referred to [HL82b] for further understanding of calibrated geometry and to [Mor08] for an overview of geometric measure theory.

Definition 2.1. Let ϕ be a smooth m -form on a Riemannian manifold (X, g) . At a point $x \in X$ we define the **comass** of ϕ_x to be

$$\|\phi\|_{x,g}^* = \max \{ \phi_x(\vec{V}_x) : \vec{V}_x \text{ is a unit simple } m\text{-vector at } x \}$$

where "simple" means $\vec{V}_x = e_1 \wedge e_2 \cdots \wedge e_m$ for some $e_i \in T_x X$.

Remark 2.2. $\|\phi\|_g^*$ will be viewed as a pointwise function in this paper. Generally it is merely continuous. At a point x where $\phi_x \neq 0$,

$$\begin{aligned} \|\phi\|_{x,g}^* &= \max \{ \phi(\vec{V}_x) : \vec{V}_x \text{ is a simple } m\text{-vector at } x \text{ with } \|\vec{V}_x\|_g = 1 \} \\ &= \max \{ 1/\|\vec{V}_x\|_g : \vec{V}_x \text{ is a simple } m\text{-vector at } x \text{ with } \phi(\vec{V}_x) = 1 \} \\ &= 1/\min \{ \|\vec{V}_x\|_g : \vec{V}_x \text{ is a simple } m\text{-vector at } x \text{ with } \phi(\vec{V}_x) = 1 \}. \end{aligned}$$

Definition 2.3. Let $(\mathcal{E}'_*(X), d)$ denote the dual complex of the de Rham complex of X . Elements of $\mathcal{E}'_k(X)$ are k -dimensional de Rham **currents** (with compact support) and d is the adjoint of exterior differentiation.

Definition 2.4. Let T be a de Rham m -current in (X, g) . The **mass** of T is $\mathbf{M}(T) = \sup\{T(\psi) : \psi \text{ smooth } m\text{-form with } \sup_X \|\psi\|_g^* \leq 1\}$.

When $\mathbf{M}(T) < \infty$, T determines a unique Radon measure $\|T\|$ characterized by $\int_X f \cdot d\|T\| = \sup\{T(\psi) : \|\psi\|_{x,g}^* \leq f(x)\}$ for any nonnegative continuous function f on X . Therefore $\mathbf{M}(T) = \|T\|(X)$. Moreover, the Radon-Nikodym Theorem asserts the existence of a $\|T\|$ measurable tangent m -vector field \vec{T} a.e. with vectors $\vec{T}_x \in \Lambda^m T_x X$ of unit length in the dual norm of the comass norm, satisfying

$$(2.1) \quad T(\psi) = \int_X \psi_x(\vec{T}_x) d\|T\|(x) \text{ for any smooth } m\text{-form } \psi,$$

or briefly $T = \vec{T} \cdot \|T\|$ a.e. $\|T\|$.

When T enjoys local finite mass, one can get Radon measure $\|T\|$ and decomposition (2.1) as well.

Definition 2.5. Let $\text{spt}(f)$ be the support of f where f is a function. For a current T , let U_T stand for the largest open set with $\|T\|(U_T) = 0$. Then the support of T is denoted by $\text{spt}(T) = U_T^c$.

Definition 2.6. Let $\mathbb{M}_k(X) = \{T \in \mathcal{E}'_k(X) : \mathbf{M}(T) < \infty\}$. Then $N_k(X) = \{T \in \mathbb{M}_k(X) : dT \in \mathbb{M}_{k-1}(X)\}$ is the space of k -dimensional **normal** currents.

Remark 2.7. We view a current in \mathbb{M}_k as a functional over smooth k -forms not a specific representative of generalized distribution.

Note that $(N_*(X), d)$ form a complex. Recalling the natural isomorphisms established by de Rham and Federer and Fleming:

$$H_*(\mathcal{E}'_*(X)) \cong H_*(X; \mathbb{R}) \cong H_*(N_*(X))$$

we identify these three homology groups the same.

Definition 2.8. A smooth form ϕ on (X, g) is called a **calibration** if $\sup_X \|\phi\|_g^* = 1$ and $d\phi = 0$. The triple (X, ϕ, g) is called a **calibrated manifold**. If M is an oriented (singular) submanifold with $\phi|_M$ equal to the volume form of M , then (ϕ, g) is a **calibration pair** of M on X . We say ϕ **calibrates** M and M **can be calibrated** in (X, g) .

Definition 2.9. Let ϕ be a calibration on (X, g) . We say that a current T of local finite mass is **calibrated** by ϕ , if $\phi_x(\vec{T}_x) = 1$ a.a. $x \in X$ for $\|T\|$.

The following is the fundamental theorem of calibrated geometry in [HL82b].

Theorem 2.10 (Harvey and Lawson). *If T is a calibrated current with compact support in (X, ϕ, g) and T' is any compactly supported current homologous to T (i.e., $T - T'$ is a boundary and in particular $dT = dT'$), then*

$$\mathbf{M}(T) \leq \mathbf{M}(T')$$

with equality if and only if T' is calibrated as well.

Remark 2.11. *Let M be an oriented compact smooth submanifold. Then by continuity method, the current $[[M]] = \int_M \cdot$ is calibrated if and only if M is calibrated.*

We shall use certain properties of comass. Especially, Lemma 2.15 is crucial to our methods and Lemma 2.14 gives a concrete control of comass for metric gluing.

Lemma 2.12. *For any metric g , m -form ϕ and positive function f on X ,*

$$\|\phi\|_{f \cdot g}^* = f^{-\frac{m}{2}} \cdot \|\phi\|_g^*.$$

Proof. By the formula in Remark 2.2. ■

Lemma 2.13. *For any m -form ϕ and metrics $g' \geq g$ on X , we have*

$$\|\phi\|_{g'}^* \leq \|\phi\|_g^*.$$

Proof. By the definition of comass. ■

Lemma 2.14 (Comass Control for Gluing Procedure). *For any m -form ϕ , positive functions a and b , and metrics g_1 and g_2 , it follows*

$$(2.2) \quad \|\phi\|_{ag_1+bg_2}^* \leq \frac{1}{\sqrt{a^m \cdot \frac{1}{\|\phi\|_{g_1}^{*2}} + b^m \cdot \frac{1}{\|\phi\|_{g_2}^{*2}}}}$$

where $\frac{1}{0}$ and $\frac{1}{+\infty}$ are identified with $+\infty$ and 0 respectively.

Proof. The statement is trivial where $\phi = 0$. Now consider a point x at where $\phi_x \neq 0$. In the subspace spanned by a simple m -vector \vec{V}_x , there exists an orthonormal basis (e_1, \dots, e_m) of g_1 , under which g_2 is diagonalized as $\text{diag}(\lambda_1, \dots, \lambda_m)$ for some $\lambda_i > 0$. Suppose $\vec{V}_x = te_1 \wedge \dots \wedge e_m$, then

$$(2.3) \quad \begin{aligned} \|\vec{V}_x\|_{ag_1+bg_2}^2 &= t^2(a + b\lambda_1) \cdots (a + b\lambda_m) \\ &= t^2[a^m + \cdots + b^m \prod \lambda_i] \\ &\geq t^2 a^m + t^2 b^m \prod \lambda_i \\ &= a^m \|\vec{V}_x\|_{g_1}^2 + b^m \|\vec{V}_x\|_{g_2}^2. \end{aligned}$$

By Remark 2.2, (2.3) implies (2.2). ■

Lemma 2.15 (Comass One Lemma). *Suppose (E, π) is a disc bundle over M (as the zero section) and g is a Riemannian metric defined on E . Then each fiber is perpendicular to M if and only if $\pi^* \text{vol}_{g_M}$ has comass one pointwise along M where vol_{g_M} means the volume form of M induced by g .*

Proof. Fix a point x on M . Take an oriented orthonormal basis $\{e_1, \dots, e_m\}$ of $T_x M$. Then we have unique decompositions $e_i = \sin \theta_i \cdot a_i + \cos \theta_i \cdot b_i$ where a_i is a unit vector perpendicular to

F_x – the subspace of the fiber directions in $T_x E$, b_i is some unit vector in F_x , and θ_i is the angle between e_i and F_x for $i = 1, \dots, m$. Denote vol_{g_M} by ω . By the choice of $\{e_i\}$,

$$\begin{aligned}
 1 &= \omega(e_1 \wedge e_2 \cdots \wedge e_m) \\
 &= \pi^* \omega(e_1 \wedge e_2 \cdots \wedge e_m) \\
 &= \pi^* \omega(\sin \theta_1 \cdot a_1 \wedge \cdots \wedge \sin \theta_m \cdot a_m) \\
 &= \prod \sin \theta_i \cdot \pi^* \omega(a_1 \wedge a_2 \cdots \wedge a_m).
 \end{aligned}
 \tag{2.4}$$

The third equality is due to the fact that elements of F_x annihilate $\pi^* \omega$. Since $\{a_i\}$ are of unit length, $\|\pi^* \omega\|_{x,g}^* \geq 1$, $\forall x \in M$. Combined with Remark 2.2, the equality holds if and only if $F_x \perp T_x M$. ■

Since the pullback of volume form is simple and well behaves, we want to extend its certain multiple to a global potential calibration form. In order to proceed, we need global forms.

2.2. For Global Forms. In the singular homology theory the *Kronecker* product $\langle \cdot, \cdot \rangle$ between cochains and chains induces a homomorphism

$$\kappa : H^q(X; G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_q(X; \mathbb{Z}), G) \text{ given by}$$

$$\kappa([z^q])([z_q]) \triangleq \langle [z^q], [z_q] \rangle$$

where G is any Abelian group. A classical result asserts that κ is surjective. When $G = \mathbb{R}$, by *de Rham* Theorem, $\kappa : H_{dR}^q(X) \xrightarrow{\sim} \text{Hom}_{\mathbb{R}}(H_q(X; \mathbb{R}), \mathbb{R})$.

Suppose $\{M_\alpha\}$ are mutually disjoint m -dimensional oriented connected compact submanifolds with the represented homology classes $\{[M_\alpha]\}$ lying in one common side of some hyperplane through the zero of $H_m(X; \mathbb{R})$. Then there exists a homomorphism $F \in \text{Hom}_{\mathbb{R}}(H_m(X; \mathbb{R}), \mathbb{R})$ forwarding $\{[M_\alpha]\}$ to positive numbers. Consequently, we have the following existence result.

Lemma 2.16. *Suppose X is a manifold and $\{M_\alpha\}$ satisfy the above condition. Then there exists a closed m -form ϕ on X with $\int_{M_\alpha} \phi > 0$ for every M_α .*

Remark 2.17. *When $\#\{M_\alpha\} = r < \infty$, the requirement becomes the **convex hull condition** that $\{\sum_{i=1}^r t_i [M_i] : \sum_{i=1}^r t_i = 1 \text{ and } t_i \geq 0\}$ in $H_m(X; \mathbb{R})$ does not contain the zero class.*

2.3. Bundle Structure around Submanifolds. Given an oriented compact submanifold M in (X, g) , consider its ϵ -neighborhood U_ϵ . When ϵ is small enough, the metric induces a disc-fibered bundle structure of U_ϵ , whose fiber is given by the exponential map restricted to normal directions along M . We call the orthogonal complement to fiber directions in TU_ϵ **horizontal directions** and a **horizontal change** means a smoothly varying modification on g along horizontal directions.

By a strong deformation retraction from U_ϵ to M , $H^m(U_\epsilon; \mathbb{R}) \cong H^m(M; \mathbb{R})$. Therefore for any $[\phi_1]$ and $[\phi_2] \in H^m(U_\epsilon; \mathbb{R})$

$$[\phi_1] = [\phi_2] \Leftrightarrow \int_M \phi_1 = \int_M \phi_2. \tag{2.5}$$

For an oriented properly embedded non-compact complete submanifold (without boundary), similar bundle structure occurs for some suitable smooth positive function ϵ defined along the submanifold.

Besides a manifold, a calibrated manifold consists of a balanced pair of a metric and a closed form. We shall do gluing procedures in both slots.

2.4. Gluing of Forms. Suppose M is an oriented compact connected m -dimensional submanifold with $[M] \neq [0] \in H_m(X; \mathbb{R})$. By §2.2 there exists a closed m -form ϕ on X with $s = \int_M \phi > 0$.

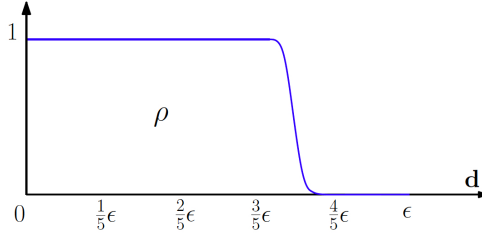
Let π_g be the projection map of the disc bundle in §2.3. Then in U_ϵ

$$(2.6) \quad \int_M \frac{s \cdot \pi_g^* \omega_M}{\text{Vol}_g(M)} = s = \int_M \phi$$

where ω_M is the volume form of M . Denote the integrand of the left hand side of (2.6) by ω^* . By (2.5) $[\omega^*] = [\phi]$ in $H^m(U_\epsilon; \mathbb{R})$ which indicates

$$(2.7) \quad \phi = \omega^* + d\psi$$

for some smooth $(m-1)$ -form ψ defined on U_ϵ . Now take $\Phi = \omega^* + d((1 - \rho(\mathbf{d}))\psi)$ where \mathbf{d} is the distance function to M and ρ is given in the picture.



Clearly Φ extends to a closed smooth form on X as follows.

$$\Phi = \begin{cases} \omega^* & 0 \leq \mathbf{d} \leq \frac{3}{5}\epsilon \\ \omega^* + d(1 - \rho(\mathbf{d}))\psi & \frac{3}{5}\epsilon < \mathbf{d} \leq \frac{4}{5}\epsilon \\ \phi & \frac{4}{5}\epsilon < \mathbf{d} \end{cases}$$

By Lemma 2.15, $\|\Phi\|_g^* = \|\omega^*\|_g^* = \frac{s}{\text{Vol}_g(M)}$ along M .

3. HORIZONTAL CHANGE OF METRIC

3.1. Gluing of Metrics. In this subsection we glue metrics for the horizontal change. By the bundle structure of U_ϵ , we define

$$\bar{g} = \left(\frac{s}{\text{Vol}_g(M)} \right)^{\frac{2}{m}} \pi_g^*(g_M) \oplus g_F$$

where g_F is any smooth metric along fiber directions and $\pi_g^*(g_M)(v_q, v'_q) = g_M(\pi_*^g(v_q), \pi_*^g(v'_q))$ where π_*^g is the push-forwarding map of π_g .

Proposition 3.1. $\|\omega^*\|_{\bar{g}}^* = 1$ on U_ϵ .

Proof. Fix a point q in U_ϵ . By Remark 2.2 one only needs to verify that

$$(3.1) \quad \min\{\|W\|_{\bar{g}} : W \text{ is a simple } m\text{-vector at } q \text{ with } \omega^*(W) = 1\}$$

equals one. Suppose \bar{W} realizes the minimum of (3.1) and it decomposes into a purely horizontal part and a fiber-involving part: $\bar{W} = \bar{W}^h + \bar{W}^v$. By Lemma 2.12 and definitions of ω^* and \bar{g} ,

$$\begin{aligned} 1 = \omega^*(\bar{W}) &= \frac{s\pi_g^*\omega_M(\bar{W})}{\text{Vol}_g(M)} = \frac{s\pi_g^*\omega_M(\bar{W}^h)}{\text{Vol}_g(M)} = \frac{s\omega(\pi_*^g\bar{W}^h)}{\text{Vol}_g(M)} \\ &= \frac{s\|\pi_*^g\bar{W}^h\|_{g_M}}{\text{Vol}_g(M)} = \|\pi_*^g\bar{W}^h\|_{(\frac{s}{\text{Vol}_g(M)})^{\frac{2}{m}}g_M} = \|\bar{W}^h\|_{\bar{g}} \leq \|\bar{W}\|_{\bar{g}}. \end{aligned}$$

The equality holds if and only if \bar{W} is purely horizontal. Hence (3.1) is indeed one. ■

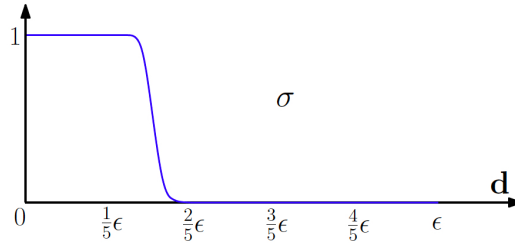
Remark 3.2. Due to the dimension reason, \bar{W} is the unique oriented horizontal m -vector with $\omega^*(\bar{W}) = 1$. In particular $\|\omega^*\|_{\bar{g}}^* = \frac{1}{\|\bar{W}\|_{\bar{g}}}$ for any metric.

We construct a new metric $\tilde{g} = \tilde{g}^h \oplus \tilde{g}^v$ based on $g = g^h \oplus g^v$ by gluing metrics along horizontal and fiber directions respectively:

$$(3.2) \quad \tilde{g}^h = \sigma^{\frac{1}{m}} \left(\left(\frac{s}{\text{Vol}_g(M)} \right)^{\frac{2}{m}} + \mathbf{d}^2 \right) \pi_g^*(g_M) + (1 - \sigma)^{\frac{1}{m}} \alpha g^h, \text{ and}$$

$$(3.3) \quad \tilde{g}^v = \sigma g_F + (1 - \sigma) \alpha g^v,$$

where \mathbf{d} is as previous, α is a smooth positive function (to be determined below) on X , and $\sigma = \sigma(\mathbf{d})$ is given in the picture.



Let g_F be αg^v . Then $\tilde{g}^v = \alpha g^v$. Although expressions (3.2) and (3.3) are valid in U_ϵ only, \tilde{g} can be extended to a global metric by defining $\tilde{g} = \alpha g$ on $(U_\epsilon)^c$. Choose an appropriate α such that

$$(3.4) \quad \|\Phi\|_{\alpha g}^* < 1 \text{ on } X,$$

which implies

$$(3.5) \quad \alpha^m \|\bar{W}\|_g^2 > 1 \text{ on } U_{\frac{3}{5}\epsilon}.$$

Then

$$\|\Phi\|_{\tilde{g}}^* = \begin{cases} 1 & 0 = \mathbf{d} \\ \|\omega^*\|_{\tilde{g}}^* < \|\omega^*\|_{\tilde{g}}^* = 1 & 0 < \mathbf{d} \leq \frac{1}{5}\epsilon \\ \|\omega^*\|_{\tilde{g}}^* = \|\omega^*\|_{\tilde{g}^h \oplus \tilde{g}^v}^* & \\ \leq \frac{1}{\sqrt{\sigma\|\bar{W}\|^2 + \alpha^m(1-\sigma)\|\bar{W}\|_{g^h}^2}} & \frac{1}{5}\epsilon \leq \mathbf{d} \leq \frac{2}{5}\epsilon \\ & \sqrt{\frac{\sigma\|\bar{W}\|^2}{(\frac{s}{\text{Vol}_g(M)})^{\frac{2}{m}} + \mathbf{d}^2 \mu_g^*(g_M)}}} \\ < 1 & \\ \|\Phi\|_{\alpha g}^* < 1 & \frac{2}{5}\epsilon \leq \mathbf{d} \end{cases}$$

The first inequality is by Proposition 3.1, the second from Lemma 2.14 and Remark 3.2, the third by Proposition 3.1 and (3.5), and the last due to (3.4). In summary, we obtain a calibration pair (Φ, \tilde{g}) with $\mathbf{spt}(\|\Phi\|_{\tilde{g}}^* - 1) = M$.

Remark 3.3. If X is compact, then α can be chosen as a sufficiently large constant to guarantee (3.4). It is clear that the same constant α still works when one shrinks the neighborhood U_ϵ .

3.2. Compact X and Connected M .

Definition 3.4. A submanifold M is **strongly calibrated** (or **tamed**) in a calibrated manifold (X, ϕ, g) , if each connected component of M is calibrated by ϕ (either ϕ or $-\phi$) and $\mathbf{spt}(\|\Phi\|_g^* - 1) = M$.

Theorem 3.5. Suppose (X, g) is a compact Riemannian manifold and M is an oriented compact connected m -dimensional submanifold representing a nonzero class of $H_m(X; \mathbb{R})$. Then for any open neighborhood U of M , a new metric \hat{g} can be constructed by a horizontal change of g supported in U such that M is strongly calibrated by some calibration $\hat{\Phi}$ in (X, \hat{g}) .

Proof. By the compactness of M , there exists a small positive number ϵ to guarantee the disc bundle structure of U_ϵ in §2.3 and $U_\epsilon \subseteq U$. By §3.1 and Remark 3.3, a new metric \tilde{g} can be constructed by a horizontal change in U_ϵ with $\tilde{g} = \alpha g$ on $X - U_\epsilon$, where α is a large constant satisfying (3.4). Define $\hat{g} \triangleq \alpha^{-1}\tilde{g}$ and $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}}\Phi$. By Lemma 2.12, M is strongly calibrated in $(X, \hat{\Phi}, \hat{g})$. Moreover \hat{g} equals g on $X - U_\epsilon$. ■

In fact metrics produced in the proof make M more than being minimal.

Proposition 3.6. M is totally geodesic under the constructed metric \hat{g} .

Proof. Note that the projection map is length-shrinking. ■

3.3. Compact X and Non-Connected M .

Definition 3.7. A family \mathfrak{M} of mutually disjoint connected oriented compact submanifolds of a manifold X (not necessarily compact) is called a **mutually disjoint collection** and an element of \mathfrak{M} is a **component**. The (nonempty) subset \mathfrak{M}_k of all components of dimension k is its **k -level**. When \mathfrak{M} consists of finitely (or countably) many components, we call it a **finite (or countable) collection**. If \mathfrak{M} has only one level in dimension m , then it is called an **m -collection**. Let $\underline{\mathfrak{M}}_k$ denote the union submanifold of the components of \mathfrak{M}_k and define $\underline{\mathfrak{M}} \triangleq \bigcup_k \underline{\mathfrak{M}}_k$ as a set.

In this subsection we assume that X is compact.

Theorem 3.8. Let $\mathfrak{M} = \{M_i\}_{i=1}^s$ be a finite mutually disjoint m -collection satisfying the convex hull condition. Then for any open neighborhood U of $\underline{\mathfrak{M}}$, a new metric \hat{g} can be constructed by a horizontal change of g supported in U such that there exists a calibration m -form $\hat{\Phi}$ on (X, \hat{g}) and every nonzero current $T = \sum_{i=1}^s t_i [[M_i]]$ with $t_i \geq 0$ is calibrated in $(X, \hat{\Phi}, \hat{g})$. Consequently T is mass-minimizing in $[T]$ with $\mathbf{M}(T) = \sum_{i=1}^s t_i \text{Vol}_{\hat{g}}(M_i)$.

If we just require that each component represents a nonzero class, then there exists some hyperplane \mathcal{P}_m through zero in $H_m(X; \mathbb{R})$ avoiding all classes $\{[M_i]\}_1^s$. \mathcal{P}_m divides the space into two open chambers. By reversing orientations of components of \mathfrak{M}_m in one chamber, we get a new collection satisfying the convex hull condition.

Corollary 3.9. Suppose \mathfrak{M} is a finite mutually disjoint m -collection in (X, g) and each component represents a nonzero class in $H_m(X; \mathbb{R})$. Then for any open neighborhood U of $\underline{\mathfrak{M}}$, a new metric \hat{g} can be constructed by a horizontal change of g supported in U such that $\underline{\mathfrak{M}}_m$ can be tamed in (X, \hat{g}) .

Proof of Theorem 3.8. By assumption there exists a smooth closed m -form ϕ on X with $\int_{M_i} \phi > 0$ for every M_i . Since \mathfrak{M}_m is a finite collection, one can choose $\epsilon > 0$ such that $\{U_\epsilon(M_i)\}$ are mutually disjoint with bundle structures in §2.3 and contained in U . Gluing ϕ with local forms as in §2.4 on $\{U_\epsilon(M_i)\}$, we get Φ . Using a large constant α satisfying (3.4) for the metric gluing, one can obtain a new metric g_α under which Φ becomes a calibration. Let $\hat{g} \triangleq \alpha^{-1} g_\alpha$ and $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}} \Phi$. Then every M_i is calibrated by $\hat{\Phi}$ in (X, \hat{g}) .

Since $\{M_i\}$ are submanifolds, the measure $||[[M_i]]||$ is $\mathcal{H}|_{M_i}$, the Hausdorff m -measure restricted to M_i in (X, \hat{g}) . When $T = \sum_{i=1}^s t_i [[M_i]]$ with $t_i \geq 0$, we have $\|T\| = \sum_i t_i \mathcal{H}|_{M_i}$ and therefore $\mathbf{M}(T) = \int_X \|\vec{T}\|_{\hat{g}} d\|T\| = \sum t_i \int_{M_i} d \text{vol}_{\hat{g}} = \sum t_i \text{Vol}_{\hat{g}}(M_i)$. ■

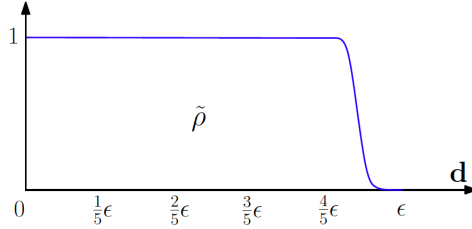
Remark 3.10. If $\sum s_i [M_i] = \sum t_j [M_j]$ for some $s_i, t_j \geq 0$, then we have $\sum s_i \text{Vol}_{\hat{g}}(M_i) = \sum t_j \text{Vol}_{\hat{g}}(M_j)$ automatically.

3.4. Compact X and \mathfrak{M} with Several Levels. Assume \mathfrak{M} is a finite mutually disjoint collection, not necessarily of a single level, in a compact Riemannian manifold (X, g) . Results similar to Theorem 3.8 and Corollary 3.9 can be gained.

Theorem 3.11. Suppose \mathfrak{M} is a finite mutually disjoint collection and each k -level \mathfrak{M}_k satisfy the convex hull condition. Then for any open neighborhood U of \mathfrak{M} , a new metric \hat{g} can be constructed by a horizontal change of g supported in U such that there exist a family of calibrations $\{\hat{\Phi}_k\}$ in (X, \hat{g}) and every nonzero current $T = \sum_{i=1}^{s_k} t_i [[M_i]]$ with $M_i \in \mathfrak{M}_k$ and $t_i \geq 0$ is calibrated by $\hat{\Phi}_k$. Consequently T is mass-minimizing in $[T]$ with $\mathbf{M}(T) = \sum_{i=1}^s t_i \text{Vol}_{\hat{g}}(M_i)$.

Corollary 3.12. Let \mathfrak{M} be a finite mutually disjoint collection in (X, g) with each component representing a nonzero class in the \mathbb{R} -homology of X . Then for any open neighborhood U of \mathfrak{M} , a new metric \hat{g} can be constructed by a horizontal change of g supported in U such that every \mathfrak{M}_k can be tamed in (X, \hat{g}) .

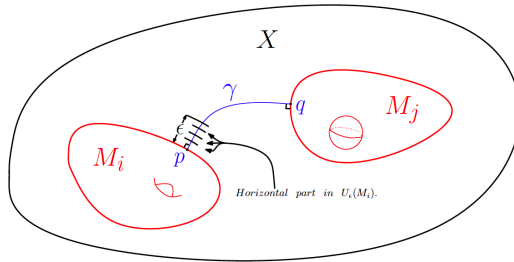
Proof of Theorem 3.11. Without loss of generality, assume $\mathfrak{M} = \{A^a, B^b\}$ with $a > b$. Take $\epsilon > 0$ to guarantee bundle structures of $U_\epsilon(A)$ and $U_\epsilon(B)$ in §2.3, $U_\epsilon(A) \cap U_\epsilon(B) = \emptyset$ and $U_\epsilon(A \cup B) \subseteq U$. Suppose one gets Φ for A as in §2.4. Then $\Phi = d\theta$ in $U_\epsilon(B)$. Set $\tilde{\Phi} = \Phi - d(\tilde{\rho}(\mathbf{d})\theta)$ where $\tilde{\rho}$ is given in the picture. Then $\tilde{\Phi}$ is zero on $U_{\frac{4}{5}\epsilon}(B)$. There exists a function α_a with value one on $\{\tilde{\rho} = 1\}$ satisfying (3.4). Based on $\tilde{\Phi}$, g and α_a , one can get a metric \tilde{g} as in §3.1 such that A is calibrated by $\tilde{\Phi}$ under \tilde{g} .



By the compactness of X , one can choose a b -form ψ with $\int_B \psi > 0$ and $\|\psi\|_g^* < 1$ on X . Then by taking $\alpha_b \equiv 1$, one can get (Ψ, \hat{g}) as in §3.1. Now $\tilde{\Phi}$ and Ψ calibrate A and B respectively in (X, \hat{g}) . ■

Remark 3.13. Another proof is to choose a proper constant factor α for each level without changing potential calibration forms. However it does not work for the conformal change in §4.

Proposition 3.14. For the resulted metric \hat{g} , $\text{dist}_{\hat{g}}(M_i, M_j) = \text{dist}_g(M_i, M_j)$.



Proof. Suppose γ is a geodesic segment realizing $\text{dist}_g(M_i, M_j)$. According to §2.3, $\gamma \cap U_\epsilon(M_i)$ is contained in some single fiber of $U_\epsilon(M_i)$ and similarly around M_j . Since \hat{g} and g are the

same along fiber directions, it follows that $\text{dist}_{\hat{g}}(M_i, M_j) \leq l_{\hat{g}}(\gamma) = l_g(\gamma) = \text{dist}_g(M_i, M_j)$ where l stands for the length functional. Appendix A shows that the foliations induced by g and \hat{g} of $U_{\epsilon}(M_i)$ coincide. Hence the same kind of argument leads to the opposite inequality. ■

3.5. Non-Compact X . We want to extend our gluing methods to the case of non-compact ambient manifolds. However in contrast to Remark 3.3 for the compact case, generally a constant function α cannot meet the need of (3.4). For example, consider the surface X obtained by rotating the graph of $y = e^x$ around the x -axis with the induced metric from Euclidean \mathbb{R}^3 . Take γ as the circle corresponding to $x = 0$. Obviously there is no way to make γ homologically minimal through a local change of metric.

Theorem 3.15. *Suppose \mathfrak{M} is a finite mutually disjoint collection and each component represents a nonzero class in the \mathbb{R} -homology of X . Then for any open neighborhood V of \mathfrak{M} with a compact closure \overline{V} , a new metric \hat{g} can be constructed by a horizontal change of g supported in V and a conformal change in $X - V$ such that every \mathfrak{M}_k can be tamed in (X, \hat{g}) .*

Proof. One can modify the proof of Theorem 3.11. based on the compactness of \overline{V} . ■

In order to apply gluing techniques to a countable mutually disjoint collection, we introduce the following definition.

Definition 3.16. *Let $\mathfrak{M} = \{M_i\}_{i=1,2,\dots}$ be a countable mutually disjoint collection in X . If the set $\bigcup_{i \neq j} M_i$ is closed for any positive integer j , then we call \mathfrak{M} a **neat mutually disjoint collection**.*

The neatness implies the existence of $\epsilon_i > 0$ such that $U_{\epsilon_i}(M_i)$ are mutually disjoint. Hence we get the following.

Theorem 3.17. *Suppose \mathfrak{M} is a neat mutually disjoint collection and each component represents a nonzero class in the \mathbb{R} -homology of X . In addition, assume every level of \mathfrak{M} consist of finite components except the lowest (nonempty) level. Then for any open neighborhood U of \mathfrak{M} , a new metric \hat{g} can be constructed by a horizontal change of g supported in U and a conformal change in the complement of the support of the horizontal change such that every \mathfrak{M}_k can be tamed in (X, \hat{g}) .*

4. CONFORMAL CHANGE OF METRIC

4.1. Parallel Results to Section 3. An interesting kind of metric changes is the conformal change, the simplest way to vary a metric. Let us first glance at a basic case to understand our gluing method for conformal change.

Theorem 4.1. *Suppose (X, g) is a compact Riemannian manifold and M is an oriented compact connected m -dimensional submanifold representing a nonzero class in $H_m(X; \mathbb{R})$. Then for any open neighborhood U of M , a new metric \hat{g} can be constructed by a conformal change of g supported in U such that M is strongly calibrated by some calibration $\hat{\Phi}$ in (X, \hat{g}) .*

Proof. Since Φ in §2.2 is pointwise a multiple of $\pi_g^* \omega$ in $U_{\frac{3}{5}\epsilon}(M)$, $\|\Phi\|_g^*$ is smooth on $U_{\frac{3}{5}\epsilon}(M)$. Let $g' = (\|\Phi\|_g^*)^{\frac{2}{m}} g$. Then $\|\Phi\|_{g'}^* = 1$ on $U_{\frac{3}{5}\epsilon}(M)$. Take

$$(4.1) \quad \tilde{g} = \sigma^{\frac{1}{m}}(1 + \mathbf{d}^2)g' + \alpha(1 - \sigma)^{\frac{1}{m}}g$$

where σ and \mathbf{d} are the same as in §3.1 and α is a large constant satisfying (3.4). Set $\hat{g} \triangleq \alpha^{-1}\tilde{g}$ and $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}}\Phi$. Then M is strongly calibrated by $\hat{\Phi}$ under \hat{g} . Furthermore, \hat{g} is conformal to g and $\hat{g} = g$ on $X - U_\epsilon(M)$. \blacksquare

The same local gluing ideas and elimination tricks on calibrations lead to the following results.

Theorem 4.2. *Suppose \mathfrak{M} is a finite mutually disjoint collection in a compact Riemannian manifold (X, g) and every nonempty level \mathfrak{M}_k satisfy the convex hull condition. Then for any open neighborhood U of \mathfrak{M} , a new metric \hat{g} can be constructed by a conformal change of g supported in U such that there exist a family of calibrations $\{\hat{\Phi}_k\}$ in (X, \hat{g}) and every nonzero current $T = \sum_{i=1}^{r_k} t_i [[M_i]]$ with $M_i \in \mathfrak{M}_k$ and $t_i \geq 0$ is calibrated by $\hat{\Phi}_k$. Consequently T is mass-minimizing in $[T]$ with $\mathbf{M}(T) = \sum_{i=1}^s t_i \text{Vol}_{\hat{g}}(M_i)$.*

Corollary 4.3. *Suppose \mathfrak{M} is a finite mutually disjoint collection in a compact Riemannian manifold (X, g) and each component represents a nonzero class in the \mathbb{R} -homology of X . Then for any open neighborhood U of \mathfrak{M} , a new metric \hat{g} can be constructed by a conformal change of g supported in U such that each \mathfrak{M}_k can be tamed in (X, \hat{g}) .*

When X is non-compact, one can obtain better results compared with Theorems 3.15 and 3.17.

Theorem 4.4. *Suppose \mathfrak{M} is a finite mutually disjoint collection in (X, g) and each component represents a nonzero class in the \mathbb{R} -homology of X . Then a new metric \hat{g} can be constructed by a conformal change of g such that every \mathfrak{M}_k can be tamed in (X, \hat{g}) .*

Theorem 4.5. *Suppose \mathfrak{M} is a neat mutually disjoint collection in (X, g) and each component represents a nonzero class in the \mathbb{R} -homology of X . In addition, assume every level of \mathfrak{M} consist of finite components except the lowest level. Then a new metric \hat{g} can be constructed by a conformal change of g such that each \mathfrak{M}_k can be tamed in (X, \hat{g}) .*

Remark 4.6. *If (X, g) is hermitian with an (almost) complex J , so are the resulted metrics.*

As an application, we strengthen Tasaki's "equivariant" theorem in [Tas93].

Theorem 4.7 (Tasaki). *Let K be a compact connected Lie transformation group of a manifold X and M be a (connected) compact oriented submanifold in X . Assume M is invariant under the action of K and it represents a nonzero \mathbb{R} -homology class of X . Then there exists a K -invariant Riemannian metric g on X such that M is mass-minimizing in homology class with respect to g .*

By our method, one can get the following from the proof of Theorem 4.9.

Theorem 4.8. *Let K be a compact Lie transformation group of a manifold X and M be a compact connected oriented submanifold in X . Assume M is invariant under the action of K and the action is orientation preserving. Then for any K -invariant Riemannian metric g^K , there exists a K -invariant metric \hat{g}^K conformal to g^K such that M can be calibrated in (X, \hat{g}^K) .*

K 's being connected can lead to another generalization.

Theorem 4.9. *Suppose \mathfrak{M} is a neat mutually disjoint collection with only the lowest level possibly consisting of infinite components, and that each component represents a nonzero class in the \mathbb{R} -homology of X . Let K be a compact connected Lie transformation group of X . Assume \mathfrak{M} is invariant under the action of K . Then for any K -invariant Riemannian metric g^K , there exists a K -invariant metric \hat{g}^K conformal to g^K such that every \mathfrak{M}_k can be tamed in (X, \hat{g}^K) .*

Proof. Without loss of generality, one only needs to consider the case of a single level. Since K is compact, there is a Haar-measure $d\mu$ with $\int_K d\mu = 1$. One can use $d\mu$ to average (2.7) for a K -invariant Φ . (Note that ω^* and \mathbf{d} are K -invariant.) Then average the corresponding α . By (4.1) one can get a K -invariant calibration pair (Φ, \hat{g}^K) . ■

4.2. On Mean Curvature Vector Fields. Let us take a short digression about mean curvature vector fields. By local calibrations, we have the following.

Corollary 4.10. *Suppose M is an oriented compact submanifold in (X, g) . Then there exists \hat{g} conformal to g such that M is minimal in (X, \hat{g}) .*

Remark 4.11. *Since either local orientation of a submanifold leads to the same metric by our method and being minimal is really a local property, the orientability and compactness requirements can be removed.*

What is more, by a direct computation, a concrete relation between mean curvature vector fields through a conformal change can be given explicitly.

Proposition 4.12. *Let M be an m -dimensional submanifold in (X, g) and $\tilde{g} = f \cdot g$ where f is a positive function. Then at a point $p \in M$,*

$$(4.2) \quad f(p) \cdot \tilde{H}_p = H_p - \frac{m}{2f(p)} \cdot \text{grad}_{g,p}^\perp(f).$$

Here H and \tilde{H} are mean vector fields of M under g and \tilde{g} respectively and $\text{grad}_g^\perp(\cdot)$ stands for the normal part of $\text{grad}_g(\cdot)$ along M .

Remark 4.13. *M can be realized totally geodesic via a conformal change if and only if M is pointwise totally umbilical.*

Theorem 4.14. *For a submanifold M (not necessarily oriented or compact) in (X, g) and any (smooth) section ξ of the normal bundle over M , there exists some metric \tilde{g} conformal to g such that $\tilde{H} = \xi$.*

Proof. Suppose the ϵ -neighborhood U_ϵ of M for some suitable positive function ϵ on M can be identified with the normal ϵ -disc bundle \mathcal{B} of M via the exponential map restricted to normal directions. Consider the smooth function f on \mathcal{B} by $f_x(y) = 1 - \frac{2}{m} \langle \xi_x - H_g, y \rangle_{g_x^\perp}$ where x is a point of M and y lies in the ϵ -disc fiber through x . Let F be the induced positive (shrink ϵ if needed) function on U_ϵ . Take $\tilde{g} = F \cdot g$. Since the differential of the identification map along M is identity, by (4.2) $H_{\tilde{g}} = H_g - \frac{m}{2} \cdot \text{grad}_g^\perp F = H_g + \frac{m}{2} \cdot \frac{2}{m} \cdot (\xi_x - H_g) = \xi_x$. ■

4.3. Non-Compact M . Now we consider non-compact submanifolds. Obviously the mass of a non-compact submanifold usually reaches infinity. In order to get adapted to this difference, we introduce the following notion.

Definition 4.15. *The (strong) Global Plateau Property of a properly embedded complete non-compact submanifold M means that for any oriented bounded domain Ω on M with $d[[\Omega]] \neq 0$ and $\mathbf{M}(d[[\Omega]]) < \infty$, $[[\Omega]]$ is a (the unique) area-minimizer among all m -dimensional rectifiable currents (see [Fed69] or [Mor08]) with boundary $d[[\Omega]]$.*

Theorem 4.16. *Suppose M^m is a properly embedded oriented connected non-compact complete submanifold without boundary in (X, g) . Then for any open neighborhood U of M , a new metric \hat{g} can be constructed via a conformal change of g supported in U such that M can be strongly calibrated with strong Global Plateau Property in (X, \hat{g}) .*

Proof. Let $U_\epsilon(M)$ be an ϵ -neighborhood of M (for some suitable positive function ϵ along M) with the bundle structure in §2.3. Set $\phi = d(\rho(\mathbf{d})\pi_g^*(\psi))$ where $d\psi$ is the volume form on M , \mathbf{d} and ρ are similar to those in §2.4. Now $\|\phi\|_g^*$ is smooth on $U_{\frac{3}{5}\epsilon}(M)$. Let $\bar{g} = (\|\phi\|_g^*)^2 g$. Then $\|\phi\|_{\bar{g}}^* = 1$ on $U_{\frac{3}{5}\epsilon}(M)$. Define $\hat{g} = \sigma^{\frac{1}{m}}(1 + (\mathbf{d})^2)\bar{g} + \alpha(1 - \sigma)^{\frac{1}{m}}g$ where α is a smooth function with value one on U_ϵ^c such that $\|\phi\|_{\alpha g}^* \leq 1$ on X , and σ is the same as in §3.1. By Lemma 2.14 and $\mathbf{spt}(\|\phi\|_{\hat{g}}^* - 1) = M$, ϕ strongly calibrates M in (X, \hat{g}) . Let Ω be an oriented bounded domain on M with $\mathbf{M}(d[[\Omega]]) < \infty$. For any competitor current K (of finite mass) with $dK = d[[\Omega]]$ we have $\mathbf{M}([[\Omega]]) = [[\Omega]](d(\rho\pi_g^*(\psi))) = (d[[\Omega]])(\rho\pi_g^*(\psi)) = K(\phi) \leq \mathbf{M}(K)$. When $K \neq [[\Omega]]$ as functionals over smooth m -forms, the inequality is strict. \blacksquare

4.4. Global Plateau Property for Compact M . How about compact submanifolds? By strong Global Plateau Property of a compact submanifold M we mean that for any domain Ω of M with $d[[\Omega]] \neq 0$ and $\mathbf{M}(d[[\Omega]]) < \infty$, $[[\Omega]]$ or (and) $[[\Omega]] - [[M]]$ is (are) the unique (two) area-minimizing rectifiable current(s) with boundary $d[[\Omega]]$.

Remark 4.17. *By approximations in Ω or Ω^c , it can be seen that the requirement $\mathbf{M}(d[[\Omega]]) < \infty$ in the definition of strong Global Plateau Property can be removed.*

Theorem 4.18. *Suppose M is an oriented compact connected m -dimensional submanifold in (X, g) and it represents a nonzero class in $H_m(X; \mathbb{R})$. Then a new metric \hat{g} can be constructed by a conformal change of g such that M can be strongly calibrated in (X, \hat{g}) with strong Global Plateau Property.*

Proof. By the compactness of M one can choose a closed smooth m -form ϕ with $\int_M \phi > 0$ and $\|\phi\|_g^* < 1$ on $\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}}$. Suppose one gets Φ based on ϕ as in §2.4. Then there exists α_ϕ with value one on $\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}}$ satisfying (3.4). Denote the resulted metric as in §3 by \hat{g} . Then $\hat{g} = g$ on $\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}}$. Assume Ω is a domain of M with finite $\mathbf{M}_{\hat{g}}(d[[\Omega]])$ and $\mathbf{M}_{\hat{g}}([[\Omega]]) < \frac{1}{2}\text{Vol}_{\hat{g}}(M)$. (The proof remains the same for the equality case.) Suppose K is a rectifiable competitor with $\mathbf{M}_{\hat{g}}(K) < \text{Vol}_{\hat{g}}(M)$ and $dK = d[[\Omega]]$. Assume also that every connected component of $\mathbf{spt}(K)$ touches $\partial\Omega$.

If $\mathbf{spt}(K) \subseteq \overline{U_\epsilon}$, then $[[\Omega]] - K$ belongs to an integer multiple of $[M]$. Therefore $\mathbf{M}_{\hat{g}}([[\Omega]]) < \mathbf{M}_{\hat{g}}(K)$ by assumption.

If $\text{spt}(K)$ is not entirely contained in $\overline{U_\epsilon}$, a slicing result asserts the existence of very small $0 < \mu < \frac{\epsilon}{30}$ such that the slice $S \triangleq K, \mathbf{d}, \epsilon - \mu >$ (corresponding to $\mathbf{d} = \epsilon - \mu$) is a rectifiable current. (Note that \mathbf{d} is Lipschitz. This can be achieved by applications of coarea formula. For example, see [Mor08].) A celebrated result in [FF60] asserts that $\{\text{integral current } T \text{ with } \mathbf{M}(T) + \mathbf{M}(dT) \leq c \text{ and } \text{spt}(T) \subseteq \Gamma\}$ is compact in the weak topology for any positive number c and compact set Γ . So there exists an area-minimizing integral current \tilde{K} with $d\tilde{K} = [[\Omega]] - S$ and $\text{spt}(\tilde{K}) \subseteq \overline{U_{\epsilon-\mu}}$. Let p' be a point in $(U_{\epsilon-3\mu} - \overline{U_{\epsilon-4\mu}}) \cap \text{spt}(\tilde{K})$. Set $\overline{K}_{\Omega, K, S, \tilde{K}}$ to be the restriction $\tilde{K} \llcorner (U_{\epsilon-\mu} - \overline{U_{\epsilon-6\mu}})$ of \tilde{K} to $U_{\epsilon-\mu} - \overline{U_{\epsilon-6\mu}}$.

Claim: There exists some positive β independent of Ω, K, S, \tilde{K} or $\hat{g}|_{X - (\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}})}$ such that $\mathbf{M}_h(\overline{K}_{\Omega, K, S, \tilde{K}}) > \beta$ for any $\overline{K}_{\Omega, K, S, \tilde{K}}$ with nonempty support where $h = g|_{U_{\epsilon-\mu} - \overline{U_{\epsilon-6\mu}}}$.

Let us require in addition at the beginning that $0 < \int_M \phi < \beta$. Then it follows by the claim that M has strong Global Plateau Property in (X, \hat{g}) . ■

Proof of Claim. The proof follows from Allard's idea. By Nash's embedding theorem [Nas56], $(\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}}, g)$ can be isometrically embedded via a map f into some Euclidean space (\mathbb{R}^s, g_E) . $\overline{K}_{\Omega, K, S, \tilde{K}}$ induces a varifold (introduced by Almgren) $V_{\overline{K}}$ supported on $U_{\epsilon-\mu} - \overline{U_{\epsilon-6\mu}}$. Since \overline{K} is a restricted area-minimizing current, $\delta V_{\overline{K}} = 0$ a.e. By the compactness of $\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}}$, the norm of the corresponding $\delta V_{f\# \overline{K}}$ in \mathbb{R}^s is bounded a.e. by a constant A related to the embedding, namely independent of \overline{K} . Note that the density of $V_{f\# \overline{K}}$ is a.e. at least one on its support. Choose a point a on $\text{spt}(f\# \overline{K}) \cap f(U_{\epsilon-3\mu} - \overline{U_{\epsilon-4\mu}})$ with this density property in \mathbb{R}^s . Take $R = \min\{\bar{\epsilon}, \tau\}$ where $\bar{\epsilon}$ is some positive number to ensure the ϵ -disc bundle structure over $f(\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}})$ in \mathbb{R}^s and $\tau = \text{dist}_{g_E}(f(U_{\epsilon-3\mu} - \overline{U_{\epsilon-4\mu}}), \partial(f(\overline{U_\epsilon} - U_{\frac{4\epsilon}{5}})))$. Then our claim follows by the following monotonicity result in [All72].

Theorem 4.19 (Allard). *Suppose $0 \leq A < \infty$, $a \in \text{support of } \|V\|$, $V \in \mathbf{V}_m(U)$, where U is an open region of \mathbb{R}^s . If $0 < R < \text{distance}(a, \mathbb{R}^s - U)$ and*

$$\|\delta V\| \mathbf{B}(a, r) \leq A \|V\| \mathbf{B}(a, r) \quad \text{whenever } 0 < r < R,$$

then $r^{-m} \|V\| \mathbf{B}(a, r) \exp Ar$ is nondecreasing in r for $0 < r < R$. ■

Remark 4.20. *By our construction, if $\text{Vol}_{\hat{g}}(\Omega) < \frac{1}{2} \text{Vol}_{\hat{g}}(M)$, then $[[\Omega]]$ is the unique of least mass among all rectifiable currents with boundary $d[[\Omega]]$.*

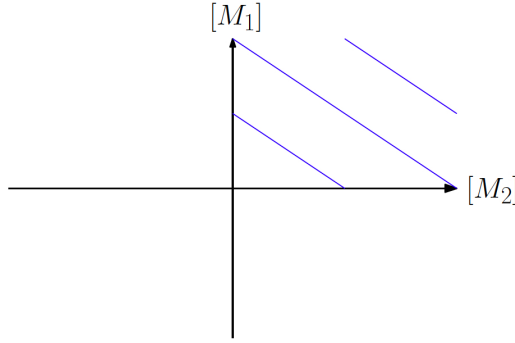
5. SEVERAL CALIBRATIONS

In Theorem 4.2, only one calibration is constructed for each dimension. Therefore it lacks control on some region in the space of homology classes. To conquer this problem we shall construct a metric which supports enough calibrations that we need for each dimension.

When X^n is oriented with betti numbers $b_k < \infty$ for $1 \leq k < \frac{1}{2}n$, by Thom [Tho54] or Corollary II.30 in [Tho07] there exist embedded oriented connected compact k -dimensional submanifolds $\mathcal{L}_k \triangleq \{M_1^k, \dots, M_{b_k}^k\}$ such that $\text{span}\{[M_i^k]\}_{i=1}^{b_k} = H_k(X; \mathbb{R})$. By transversality one can assume $\bigcup_{1 \leq k < \frac{1}{2}n} \mathcal{L}_k$ is a mutually disjoint collection.

Theorem 5.1. *Let (X^n, g) and M_i^k be given as above. Then there exists \hat{g} conformal to g such that every nonzero $\sum_{i=1}^{b_k} t_i [[M_i^k]]$ where $1 \leq k < \frac{1}{2}n$, $M_i^k \in \mathcal{L}_k$ and $t_i \in \mathbb{R}$ is the unique mass-minimizing current in $\sum_{i=1}^{b_k} t_i [M_i^k]$.*

Proof. For the sake of simplicity, assume $\dim H_k(X; \mathbb{R}) = 2$ for some $k < \frac{1}{2} \dim X$ and $\{[M_1], [M_2]\}$ is a basis where M_1 and M_2 are disjoint oriented connected compact submanifolds. Then there exist k -forms ϕ_1 and ϕ_2 on X with $\int_{M_i} \phi_j = \delta_{i,j}$. Without loss of generality, assume $\phi_1 \equiv 0$ on $U_\epsilon(M_2)$ and $\phi_2 \equiv 0$ on $U_\epsilon(M_1)$. Note that α can be chosen such that $\|\hat{\Phi}_i\|_{\hat{g}}^* \leq \frac{1}{2}$ on $(U_\epsilon(M_i))^c$ for the constructed metric \hat{g} and $\hat{\Phi}_1$ and $\hat{\Phi}_2$ in §4. A key observation is that $\pm \hat{\Phi}_1$, $\pm \hat{\Phi}_2$ and $\pm \hat{\Phi}_1 \pm \hat{\Phi}_2$ are all calibrations with respect to \hat{g} .



Obviously any nonzero linear combination of $[[M_1]]$ and $[[M_2]]$ can be calibrated in (X, \hat{g}) . For examples, those falling into the classes of the closer of the first quadrant can be calibrated by $\hat{\Phi}_1 + \hat{\Phi}_2$. The uniqueness follows as a result of $\text{spt}(\|\pm \hat{\Phi}_i\|_{\hat{g}}^* - 1) = M_i$, $\text{spt}(\|\pm \hat{\Phi}_1 \pm \hat{\Phi}_2\|_{\hat{g}}^* - 1) = \bigcup M_i$, the simpleness of $\pm \hat{\Phi}_i$ along M_i and $\pm \hat{\Phi}_1 \pm \hat{\Phi}_2$ along $M_1 \cup M_2$, and the connectedness of M_i .

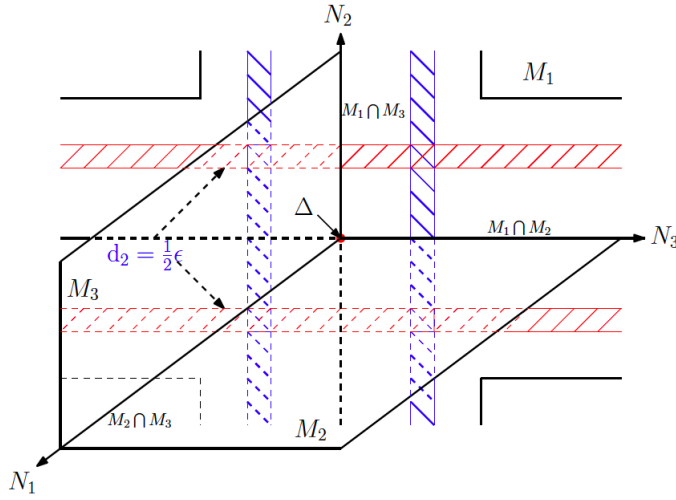
When $\dim H_k(X; \mathbb{R}) = s$, 2^s such calibrations, each of which has comass norm bounded above by $\frac{1}{s}$ away from some neighborhood of the corresponding submanifold, can be constructed for our purpose. More generally, for different dimension levels, the above argument combined with the elimination trick on calibration forms proves the theorem. ■

When the dimension of X is no less than 6, one can choose b_k smooth k -dimensional submanifolds $\{M_i^k\}_{i=1}^{b_k}$ and define $\mathcal{L}_k \triangleq \{M_1^k, \dots, M_{b_k}^k\}$ for $k = 1, 2, \dots, n-3$, such that $\text{span}\{[M_i^k]\}_{i=1}^{b_k} = H_k(X, \mathbb{R})$ and such that intersections \mathcal{I} among $\bigcup_{k=1}^{n-3} \mathcal{L}_k$ are all transversal. Note that \mathcal{I} has a natural stratification structure $\dots < \mathcal{I}_2 < \mathcal{I}_1 = \bigcup_{k=1}^{n-3} \mathcal{L}_k$, where \mathcal{I}_t is the set of intersections involving t representatives.

Theorem 5.2. *Let X^n be an oriented manifold with betti numbers $b_k < \infty$ for $1 \leq k \leq n-3$ and \mathcal{L}_k be given above. Then there exists a metric g such that every nonzero $\sum_{i=1}^{b_k} t_i [[M_i^k]]$ where $1 \leq k \leq n-3$, $M_i^k \in \mathcal{L}_k$ and $t_i \in \mathbb{R}$ is the unique mass-minimizing current in $\sum_{i=1}^{b_k} t_i [M_i^k]$.*

Proof. Choose a metric g on X such that, for any element S of \mathcal{I}_t ($t \geq 2$), there exists some $(2\epsilon\text{-cubic})$ neighborhood of S with fibers of the bundle structure in §2.3 split pointwise along S as the Riemannian product of all fibers of 2ϵ -neighborhoods of S in H_S for $H_S \in \mathcal{I}_{t-1}$ and $S \subseteq H_S$, and moreover, the horizontal part of g is the pullback of $g|_S$ via the bundle projection.

Let us focus on all the (connected parts of) deepest intersections. For simplicity, suppose we have only one connected deepest intersection $\Delta \in \mathcal{I}_3$, namely the intersection of three submanifolds M_1 , M_2 and M_3 . Assume 2ϵ is universal for $S \in \bigcup_{t \geq 2} \mathcal{I}_t$ under g in the preceding paragraph. Denote the volume forms of M_1 , M_2 and M_3 by ω_1 , ω_2 and ω_3 , and the distance functions to M_3 by \mathbf{d}_3 . \mathbf{d}_1 and \mathbf{d}_2 are defined similarly. Since $\omega_i = d\psi_i$ in the ϵ -neighborhood of $(M_i \cap M_{i+1}) \cup (M_i \cap M_{i+2})$ (subscripts in the sense of mod 3) in M_i , define $\Psi_i = d(\rho_i \psi_i)$ in the union of ϵ -cubic neighborhoods of $M_i \cap M_{i+1}$ and $M_i \cap M_{i+2}$. Here we identify the pullback of ω_i (and ψ_i) with themselves, and ρ_i is a smooth increasing function of \mathbf{d}_i with value zero when $\mathbf{d}_i \leq \frac{1}{2}\epsilon$ and one when $\frac{2}{3}\epsilon \leq \mathbf{d}_i \leq \epsilon$.



Case One: Every M_i has the same dimension k . One may want to use the form $\sum \omega_i$. However it is not well defined on the union Ξ of ϵ -cubic neighborhoods of $M_1 \cap M_2$, $M_2 \cap M_3$ and $M_3 \cap M_1$. Let us consider

$$\phi_k = \sum \omega_i - \sum \Psi_i = \sum [(1 - \rho_i)\omega_i - d\rho_i \wedge \psi_i] \text{ on } \Xi.$$

Then $\phi_k = \omega_i$ when $\mathbf{d}_i \leq \frac{1}{2}\epsilon$, $\mathbf{d}_{i+1} \geq \frac{2}{3}\epsilon$ and $\mathbf{d}_{i+2} \geq \frac{2}{3}\epsilon$.

Since $n - k \geq 3$, $\|\phi_k\|^* = \max\{\|(1 - \rho_i)\omega_i - d\rho_i \wedge \psi_i\|^*\}$ pointwise under any metric. The slash-shadow region and the backslash-shadow region are the intersections of regions $\frac{1}{2}\epsilon \leq \mathbf{d}_2 \leq \frac{2}{3}\epsilon$ and $\frac{1}{2}\epsilon \leq \mathbf{d}_3 \leq \frac{2}{3}\epsilon$ in Ξ with M_1 respectively. There are three (bunches of) directions N_i on the ϵ -cubic neighborhood $U_\epsilon(\Delta)$ of Δ . (By the choice of g , the existence of N_i cannot be guaranteed when $\mathbf{d}_i \geq 2\epsilon$.) Denote the components of g along N_i by g_i .

Choose a smooth function f_i of \mathbf{d}_i on the region $\mathbf{d}_i \leq \epsilon$ in Ξ with properties: (1) f_i equals one when $\mathbf{d}_i \geq \frac{3}{4}\epsilon$ and is strictly greater than one elsewhere, more precisely (2) $f_i = 1 + \mathbf{d}_i^2$ for $0 \leq \mathbf{d}_i \leq \frac{1}{3}\epsilon$ and (3) it has large constant value $C > 1$ such that $(\star) C^{-(n-k-2)} \|(1 - \rho_i)\omega_i - d\rho_i \wedge \psi_i\|_g^* < 1$ on each region $\Gamma_i : \frac{1}{2}\epsilon \leq \mathbf{d}_i \leq \frac{2}{3}\epsilon$ in Ξ .

For $i = 2$, modify g on $U_\epsilon(\Delta)$ in the following way:

$$g_1 \rightarrow f_2^{\rho(\mathbf{d}_1)} g_1, \quad g_3 \rightarrow f_2^{\rho(\mathbf{d}_3)} g_3, \quad \text{and} \quad g_2 \rightarrow f_2^{-\rho(\mathbf{d}_1)\rho(\mathbf{d}_3)} g_2 \quad (*)$$

where ρ is a smooth decreasing function on $[0, \epsilon]$ with value one on $[0, \frac{4}{5}\epsilon]$ and zero on $[\frac{5}{6}\epsilon, \epsilon]$. This modification extends trivially to the union Υ of $\frac{4}{5}\epsilon$ -cubic neighborhoods of $M_1 \cap M_2$, $M_2 \cap M_3$ and $M_3 \cap M_1$, and moreover, it preserves the volume form of $M_i \cap \Upsilon$.

Multiply each of g_1 , g_2 and g_3 by the product of corresponding conformal factors in $(*)$ for g and $i = 1, 2, 3$. Denote the new metric on Υ by \check{g} . The last property of f_i and $\|\omega_i - \Psi_i\|_{\check{g}}^* \leq C^{-(n-k-2)} \|\omega_i - \Psi_i\|_{\check{g}}^*$ on Γ_i in fact shows that nonzero $\sum_{i, n_i = \pm 1, 0} n_i (\omega_i - \Psi_i)$ become calibrations when restricted to (Υ, \check{g}) . Furthermore they naturally extend to calibration pairs on the union Θ of Υ and the $\frac{1}{2}\epsilon$ -neighborhood of $\bigcup M_i$ with respect to the trivial extension \bar{g} of \check{g} .

Case Two: M_2 and M_3 are of dimension k , but M_1 has a different dimension m . (Similar for the case with mutually different dimensions.) Consider potential calibrations $\pm(\omega_2 - \Psi_2)$, $\pm(\omega_3 - \Psi_3)$, $\pm(\omega_2 - \Psi_2) \pm (\omega_3 - \Psi_3)$, and $\pm(\omega_1 - \Psi_1)$ on Ξ . By the same procedure (with different weights in (\star) and $(*)$), one can get similar calibration paris in some neighborhood of $\bigcup M_i$.

Clearly the idea works for general case (with modifications in (\star) and $(*)$). Following the above procedure around all connected parts of deepest intersection, one can extend these finite preferred local calibration pairs to global ones sharing a common metric. Multiply the metric by a smooth function which is one along $\bigcup_{k=1}^{n-3} \bigcup_{M \in \mathcal{L}_k} M$ and strictly greater than one elsewhere. Name the final metric \hat{g} . By our construction, every nonzero $\sum_{i \in \mathbb{R}, M_i \in \mathcal{L}_k} t_i [[M_i]]$ with $1 \leq k \leq n-3$ can be calibrated in (X, \hat{g}) . The uniqueness of such mass-minimizing current in its current homology class follows similarly as in the proof of Theorem 5.2. Note that for any point $p \in M_i^k - \mathcal{I}_2$ the oriented unit (volume) k -vector of $\wedge^k T_p M_i^k$ is the unique among all unit k -vectors in $\wedge^k T_p X$ such that its pairing with the corresponding calibration of M_i^k produces value one. ■

Remark 5.3. For arbitrary metric h , one can still use the (local) bundle structures induce by g . For example, consider Case One above. Modify h first as follows. Let a_i denote $\|\omega_i\|_h^*$ in $U_{2\epsilon}(\Delta)$. Then multiply h along N_i by $(\frac{a_{i+1}a_{i+2}}{a_i})^{\frac{1}{n-k}}$ for $i = 1, 2, 3$. Now $\|\omega_i\|_{\tilde{h}}^* = 1$ under the new metric \tilde{h} . Based on this, one can get a metric \tilde{h} on the union of ϵ -neighborhood Σ_i of M_i (obtained by conformal changes of h along “split horizontal directions” on Σ_i) such that $\|\omega_i\|_{\tilde{h}}^* \equiv 1$ on Σ_i . (Extension can be made through $\{\epsilon < \mathbf{d}_i < 2\epsilon\} \cap U_{2\epsilon}(\Delta)$.) Then one can apply the same construction based on \tilde{h} .

Remark 5.4. The requirement of codimension no less than 3 is crucial in our proof. When $n = 4$ or 5, Theorem 5.2 can be improved to include the level of codimension 2 by [Zha].

Open Question: Usually we cannot have such existence result when k can be $n-1$. Therefore it may be interesting to study whether the same result holds for $1 \leq k \leq n-2$.

APPENDIX A. A GENERALIZED GAUSS LEMMA

Since any one-dimensional foliation is always locally orientable, one can define a local **length flow** with respect to a metric according to a choice of orientation. An observation of Sullivan [Sul78] and a special case in Harvey and Lawson [HL82a] are the following.

Theorem A.1 (Sullivan). *A one-dimensional flow is geodesible if and only if there is a transverse field of codimension one planes invariant under the length flow.*

Theorem A.2 (Harvey and Lawson). *A one-dimensional foliation Γ is geodesic if and only if its perpendicular plane field \mathcal{P} is invariant under the length flow.*

Proof. Locally, denote the oriented unit tangent vector field by V . For any local (nowhere zero) smooth section N of $\mathcal{P} \cong TX/T\Gamma$, we have $0 = N \langle V, V \rangle = 2 \langle \nabla_N V, V \rangle$. Furthermore,

$$\begin{aligned} 0 &= V \langle V, N \rangle = \langle \nabla_V V, N \rangle + \langle V, \nabla_V N \rangle \\ (A.1) \quad &= \langle \nabla_V V, N \rangle + \langle V, \nabla_N V + \mathcal{L}_V N \rangle \\ &= \langle \nabla_V V, N \rangle + \langle V, \mathcal{L}_V N \rangle. \end{aligned}$$

Hence Γ is geodesic if and only if \mathcal{P} is preserved by the local length flow. ■

Since foliations that we encounter, e.g. in the proof of Proposition 3.14, are all locally integrable, we focus on this special situation.

Corollary A.3 (Generalized Gauss Lemma). *Suppose Γ is a one-dimensional foliation and its perpendicular plane field \mathcal{P} is locally integrable. Then Γ is geodesic if and only if local integral pieces of \mathcal{P} are preserved by the local length flow along Γ .*

Remark A.4. *The appellation “Generalized Gauss Lemma” is due to the fact that one can derive Gauss Lemma from the corollary and (A.1).*

Corollary A.5. *Suppose Γ is a one-dimensional geodesic foliation and its perpendicular plane field \mathcal{P} is locally integrable. Let $g = g^\Gamma \oplus g^\perp$ be the metric decomposition of g along Γ and \mathcal{P} . Assume $\hat{g} = g^\Gamma \oplus \hat{g}^\perp$ is a smooth metric by replacing g^\perp by \hat{g}^\perp . Then Γ is geodesic as well with respect to \hat{g} .*

Proof. Since g^Γ and Γ ’s perpendicular plane field are unchanged, Γ and \hat{g} satisfy the conditions in Corollary A.3. So the conclusion follows. ■

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